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1978 J. Phys. A: Math. Gen. 11 1943

(http://iopscience.iop.org/0305-4470/11/10/012)

J. Phys. A: Math. Gen., Vol. 11, No. 10, 1978. Printed in Great Britain

Vacuum-field solutions of Ross and Sen–Dunn theories of gravitation

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Received 15 August 1977, in final form 25 April 1978

Abstract. Vacuum-field solutions of Ross and Sen–Dunn theories of gravitation have been obtained with the aid of a Friedmann-type metric. Non-static solutions are found showing that the Birkhoff theorem holds for neither theory. It has been observed that the two theories have a limited scope for vacuum solution as against the Brans–Dicke theory. Mach's principle, however, holds for both the theories.

1. Introduction

Einstein effected a geometrisation of physics in his general theory of relativity, which geometrises the phenomenon of gravitation by identifying the metric tensor of a Riemannian space–time with the gravitational potential. In the scalar–tensor theory of Brans and Dicke (1961), however, the tensor field is identified with the metric tensor of a Riemannian geometry but the scalar field remains alien to the geometry. It would, indeed, be in keeping with the spirit of Einstein's principle of geometrisation to have a scalar–tensor theory of gravitation where both the scalar and tensor fields would have geometrical significance. With this end in view two attempts have been made recently, one by Sen and Dunn (1971) and the other by Ross (1972). The Sen–Dunn theory is based on Lyra's (1951) generalisation of Riemannian geometry, while the Ross theory makes use of Weyl's modification (Weyl 1918) of the same geometry.

Recently O'Hanlon and Tupper (1972) solved the vacuum-field equations of the Brans–Dicke theory with the aid of a space–time metric of Friedmann type. They found non-static solutions showing that, in general, the Birkhoff theorem does not exist for the Brans–Dicke theory. Also they obtained solutions which may be interpreted as being contrary to Mach's principle.

It may be of some interest to examine, in the same manner, the vacuum-field solutions of the Ross and Sen–Dunn theories of gravitation with a Friedmann-type metric, which we do here in continuation of our earlier work (Krori and Nandy 1977). We have also found non-static solutions pointing to the conclusion that the Birkhoff theorem holds for neither theory. Mach's principle, however, appears to hold for both of them.

0305-4770/78/0010-1943$01.00

2. Vacuum-field equations and their solutions

2.1. Ross theory

We consider a Friedmann-type line element as follows:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\psi^2\right), \tag{1}$$

where $k = +1, 0, -1$ for a closed, flat or open space respectively.

The Ross field equations in empty space are

$$S_{\pi\beta} - \tfrac{1}{2}g_{\pi\beta}S = 0 \tag{2}$$

and

$$\phi^{|\alpha}{}_{\|\alpha} = g^{\alpha\beta}\phi_{|\alpha|\beta} - g^{\alpha\beta}\phi_{|\pi}\{\alpha\beta, \pi\} - 4\phi_{|\alpha}\phi^{|\alpha} = 0 \tag{3}$$

where

$$S_{\pi\beta} = R_{\pi\beta} - 2\phi_{|\pi|\beta} - 2\phi_{|\pi}\phi_{|\beta} + 2g_{\pi\beta}\phi^{|\alpha}\phi_{|\alpha} - g_{\pi\beta}g^{\gamma\alpha}\phi_{|\alpha|\gamma} + 2\phi_{|\alpha}\{\pi\beta, \alpha\} + g_{\pi\beta}g^{\gamma\alpha}\phi_{|\delta}\{\alpha\gamma, \delta\} \tag{4}$$

and

$$\phi^{|\alpha} \equiv g^{\alpha\beta}\phi_{|\beta}. \tag{5}$$

A double vertical bar here denotes covariant differentiation. ϕ is the fundamental scalar field in the theory. $R_{\pi\beta}$ is the usual contracted Riemann curvature tensor and $\{\alpha\gamma, \delta\}$ is the Christoffel symbol of the second kind.

With the help of equation (1), the Ross field equations (2) and (3) reduce to

$$2a\ddot{a} + \dot{a}^2 + k + a^2\dot{\phi}^2 - 2a^2\ddot{\phi} + \frac{4}{r}(1-kr^2)\phi' - 4a\dot{a}\dot{\phi} - 3(1-kr^2)\phi'^2 = 0 \tag{6}$$

$$2a\ddot{a} + \dot{a}^2 + k + a^2\dot{\phi}^2 - 2a^2\ddot{\phi} + \frac{2}{r}(1-2kr^2)\phi' - 4a\dot{a}\dot{\phi} + 2(1-kr^2)\phi'' - (1-kr^2)\phi'^2 = 0 \tag{7}$$

$$-3(k+\dot{a}^2) + 6a\dot{a}\dot{\phi} - 3a^2\dot{\phi}^2 + (1-kr^2)\phi'^2 - 2(1-kr^2)\phi'' - \frac{2}{r}(2-3kr^2)\phi' = 0 \tag{8}$$

$$\dot{\phi}' + \phi'\dot{\phi} - \frac{\dot{a}}{a}\phi' = 0 \tag{9}$$

$$(1-kr^2)\phi'' + a^2\ddot{\phi} + \frac{1}{r}(2-3kr^2)\phi' - 3a\dot{a}\dot{\phi} - 4(1-kr^2)\phi'^2 + 4a^2\dot{\phi}^2 = 0 \tag{10}$$

where primes denote differentiation with respect to r and dots denote differentiation with respect to t.

When $\phi' = 0$, i.e. $\phi = \phi(t)$ we have from equations (7), (8) and (10)

$$2a\ddot{a} + \dot{a}^2 + k + a^2\dot{\phi}^2 - 2a^2\ddot{\phi} - 4a\dot{a}\dot{\phi} = 0 \tag{11}$$

$$(\dot{a} - a\dot{\phi})^2 + k = 0 \tag{12}$$

and

$$a^2\ddot{\phi} - 3a\dot{a}\dot{\phi} + 4a^2\dot{\phi}^2 = 0. \tag{13}$$

When $\phi' \neq 0$, equation (9) can be written as

$$\frac{\dot{\phi}'}{\phi'} + \dot{\phi} - \frac{\dot{a}}{a} = 0. \tag{14}$$

Subtracting equation (7) from (6) we get

$$\frac{\phi''}{\phi'} + \phi' - \frac{1}{r(1-kr^2)} = 0. \tag{15}$$

From equations (14) and (15) we find that

$$\phi = \ln[B(t) + (2Ra/k)\sqrt{(1-kr^2)}] \tag{16}$$

where R is an arbitrary constant and $B(t)$ is an arbitrary function of time.

A straightforward calculation using equations (6)–(10) and other equations obtained from them by forming the partial derivative with regard to r leads to the following necessary conditions for any solution of equations (6)–(10) with $\phi' \neq 0$,

$$a\ddot{a} + 2k = 0, \qquad \dot{a} - a\dot{\phi} = 0, \qquad k + r^{-1}(1-kr^2)\phi' = 0. \tag{17}$$

If $\phi' \neq 0$, then $k = 0$ is excluded. If $k \neq 0$, equations (17) after insertion into equations (6)–(10) lead to a contradiction. This leaves the case $\phi' = 0$ as the only possible one for the space–times considered. Again when $\phi' = 0$, we find from equations (17) and (12) that $k = 0$ is the only possible case for any solution with $\phi' = 0$.

Now when $\phi' = 0$, we have from equations (11), (12) and (13)

$$a\ddot{a} \pm 4\dot{a}\sqrt{-k} - 4k = 0. \tag{18}$$

2.1.1. *Solution:* $k = 0$

$$ds^2 = -dt^2 + (Dt + C)^2(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\psi^2) \tag{19}$$

$$\phi = \ln[\phi_0(Dt + C)] \tag{20}$$

where ϕ_0, D, C are constants.

The following are some of the properties of the space–time (19) of this solution:

(1) For $C > 0$ and $D > 0$ the flat three-space expands indefinitely from an initial non-singular state.

(2) For $C > 0$ and $D < 0$ the initially finite flat universe contracts to a singular condition in a finite time.

In case (1), ϕ, as given by equation (20), increases indefinitely with time, whereas in case (2) it decreases without limit.

2.2. *Sen–Dunn theory*

The field equations of the Sen–Dunn theory for the combined scalar and tensor fields are

$$R_{ij} - \tfrac{1}{2}g_{ij}R = -8\pi G(x^0)^{-2}T_{ij} + \omega(x^0)^{-2}(x^0_{|i}x^0_{|j} - \tfrac{1}{2}g_{ij}x^0_{|k}x^{0|k}) \tag{21}$$

where $\omega = \frac{3}{2}$. $R_{ij} - \frac{1}{2}g_{ij}R$ is the Einstein tensor, x^0 is the scalar field, T_{ij} is the energy–momentum tensor and $x^0_{|i} = \partial x^0/\partial x^i$.

We consider a line element of the form

$$ds^2 = -(x^0)^2\, dt^2 + (x^0)^2 a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\psi^2\right). \tag{22}$$

With the aid of equation (22) the Sen–Dunn vacuum field equations may be written down as follows:

$$2a\ddot{a} + \dot{a}^2 + k + 4a\dot{a}\frac{\dot{x}^0}{x^0} + 2a^2\frac{\ddot{x}^0}{x^0} - \frac{1}{4}a^2\frac{\dot{x}^{02}}{x^{02}} - \frac{9}{4}(1-kr^2)\frac{x'^{02}}{x^{02}} - \frac{4}{r}(1-kr^2)\frac{x'^0}{x^0} = 0 \tag{23}$$

$$2a\ddot{a} + \dot{a}^2 + k + 4a\dot{a}\frac{\dot{x}^0}{x^0} + 2a^2\frac{\ddot{x}^0}{x^0} - \frac{1}{4}a^2\frac{\dot{x}^{02}}{x^{02}} - 2(1-kr^2)\frac{x''^0}{x^0}$$
$$-\frac{2}{r}(1-2kr^2)\frac{x'^0}{x^0} + \frac{1}{4}(1-kr^2)\frac{x'^{02}}{x^{02}} = 0 \tag{24}$$

$$-3(\dot{a}^2 + k) - 6a\dot{a}\frac{\dot{x}^0}{x^0} - \frac{9}{4}a^2\frac{\dot{x}^{02}}{x^{02}} + 2(1-kr^2)\frac{x''^0}{x^0} + \frac{2}{r}(2-3kr^2)\frac{x'^0}{x^0} - \frac{1}{4}(1-kr^2)\frac{x'^{02}}{x^{02}} = 0 \tag{25}$$

$$\frac{\dot{x}'^0}{x^0} - \frac{5}{4}\frac{\dot{x}^0 x'^0}{x^{02}} - \frac{\dot{a}}{a}\frac{x'^0}{x^0} = 0. \tag{26}$$

When $x'^0 = 0$, i.e. $x^0 = x^0(t)$, equations (23), (24) and (25) reduce to

$$2a\ddot{a} + \dot{a}^2 + k + 4a\dot{a}\frac{\dot{x}^0}{x^0} + 2a^2\frac{\ddot{x}^0}{x^0} - \frac{1}{4}a^2\frac{\dot{x}^{02}}{x^{02}} = 0 \tag{27}$$

and

$$\left(\dot{a} + a\frac{\dot{x}^0}{x^0}\right)^2 + k = \frac{1}{4}a^2\frac{\dot{x}^{02}}{x^{02}}. \tag{28}$$

When $x'^0 \neq 0$, equation (26) can be written as

$$\frac{\dot{x}'^0}{x'^0} - \frac{5}{4}\frac{\dot{x}^0}{x^0} - \frac{\dot{a}}{a} = 0 \tag{29}$$

and from equations (23) and (24) we have

$$\frac{x''^0}{x'^0} - \frac{5}{4}\frac{x'^0}{x^0} - \frac{1}{r(1-kr^2)} = 0. \tag{30}$$

From equations (29) and (30) we find that

$$4x^{0\,-1/4} = \left(\frac{2Aa}{k}\sqrt{(1-kr^2)} - T(t)\right) \tag{31}$$

where A is an arbitrary constant and $T(t)$ is an arbitrary function of time.

A calculation involving equations (23), (24) and (25) gives

$$2a\ddot{a} - \dot{a}^2 - k + 2a^2\frac{\ddot{x}^0}{x^0} - \frac{7}{4}a^2\frac{\dot{x}^{02}}{x^{02}} - \frac{3}{4}(1-kr^2)\frac{x'^{02}}{x^{02}} = 0. \tag{32}$$

Partial differentiation of this with respect to r gives

$$\left(\frac{\ddot{x}^0}{x^0}\right)' - \frac{7}{8}\left(\frac{\dot{x}^{02}}{x^{02}}\right)' = \frac{3}{8a^2}\left((1-kr^2)\frac{x'^{02}}{x^{02}}\right)'. \tag{33}$$

It can be seen that equation (31) does not satisfy (33) unless $x'^0 = 0$. This leaves the case $x'^0 = 0$ as the only possible one for any solution of equations (23)–(26). When $x'^0 = 0$, we find the following solution from equations (27) and (28).

2.2.1. Solution: $k = 0$

$$ds^2 = -(x^0)^2\, dt^2 + (x^0)^2[(P+1)(\beta t + N)]^{2/(P+1)}(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\psi^2) \tag{34}$$

$$x^0 = C[(P+1)(\beta t + N)]^{1/q(P+1)} \tag{35}$$

where C, β, N are constants and

$$q = -\tfrac{3}{2}, \qquad P = \tfrac{2}{3} \tag{36}$$

or

$$q = -\tfrac{1}{2}, \qquad P = -2. \tag{37}$$

Introduction of a new time coordinate τ by

$$\tau = \frac{1}{\beta P}[(P+1)(\beta t + N)]^{P/(P+1)} \tag{38}$$

in place of t shows that the solution is

$$ds^2 = C^2(\beta P\tau)^{(2/P)[1+(1/q)]}(-d\tau^2 + dx^2 + dy^2 + dz^2). \tag{39}$$

Now, both cases $q = -\frac{1}{2}$, $P = -2$ and $q = -\frac{3}{2}$, $P = \frac{2}{3}$ lead to $(2/P)[1+(1/q)] = +1$, i.e. to the same metric up to a constant factor. Obviously this metric (39) represents a conformally flat space–time.

When $k \neq 0$, no general solutions to equations (27) and (28) could be obtained. This, however, causes no difficulty in arriving at the following conclusions.

3. Conclusions

The solutions obtained above are all non-static. It therefore follows that there is no Birkhoff theorem for the Ross or the Sen–Dunn theory when the scalar field is a function of t only. This lends support to the view stated in our earlier work. O'Hanlon and Tupper also arrived at the same result in the case of the Brans–Dicke theory.

O'Hanlon and Tupper (1972) have found that some of the solutions of the Brans–Dicke theory are not in accord with Mach's principle (as usually interpreted (Brans and Dicke 1961)), because they lead to uniform space–times, whereas the corresponding values of ϕ are not uniform being functions of both r and t.

Here we find that the only possible solution (solution 2.1.1) of the Ross theory is in accord with Mach's principle. The only solution (solution 2.2.1) obtained in case of the Sen–Dunn theory is also in accord with Mach's principle. The two theories have obviously a limited scope of vacuum solution as against the Brans–Dicke theory. Both the theories, however, seem to be an improvement upon the Brans–Dicke theory with regard to Mach's principle.

Acknowledgments

The authors express their profound gratitude to the Government of Assam, Gauhati, for all facilities provided at Cotton College, Gauhati, for this piece of work. One of the authors (DN) is also grateful to Mr and Mrs A K Aditya for encouragement.

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